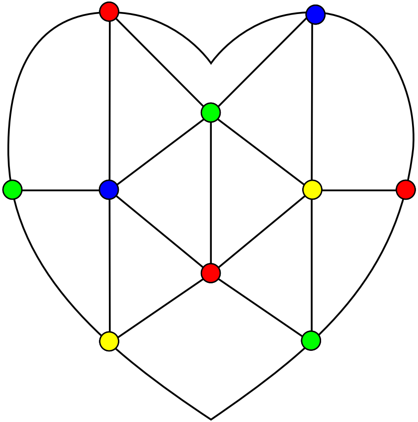
Heart of the Four Color Theorem

REDUCIBILITY

Timothy van der Valk



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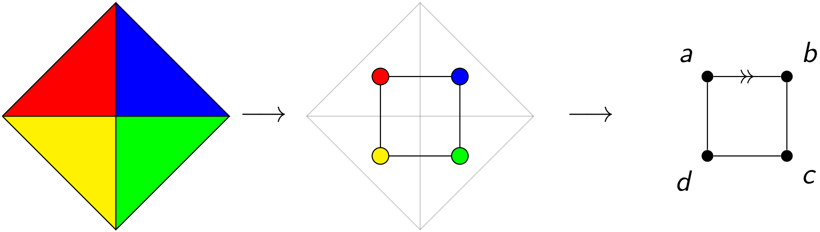
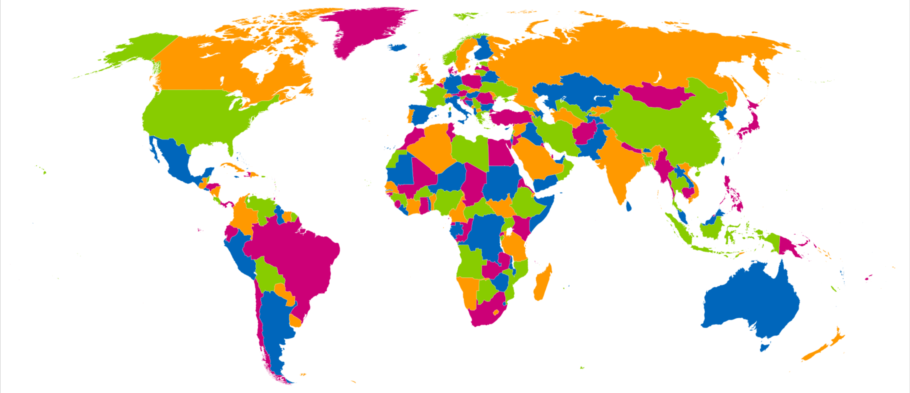
Convert a region coloring problem to a graph coloring problem.



What is a graph coloring?

Problem came forth from coloring world maps.



The Four Color Theorem

Theorem

Every planar graph can be colored in at most four colors.

A simple statement. First formulated in 1852, proven over a hundred years later in 1976.

The Five Color Theorem

Theorem

Every planar graph can be colored in at most five colors.

The Five Color Theorem

Theorem

Every planar graph can be colored in at most five colors.

Proof.

Every planar graph G has a vertex of $\deg(v) \leq 5$.

- If $\deg(v) \leq 4$, can use fifth color.
- If $\deg(v) = 5$, can always free up a color.

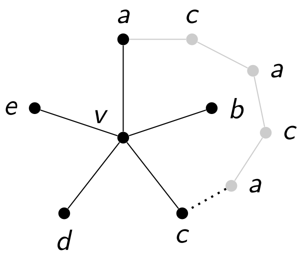
Remove v from the graph and repeat until there are no vertices left. Add back and color these vertices until we obtain a coloring of G. □

The Five Color Theorem

Most important argument

If $\deg(v) = 5$, can always free up a color.

- If there is an ac-chain, then we have isolated b. Therefore, there can not be a bd-chain from b to d. We may flip the bd-chain of b to free up the color d.
- If there is no ac-chain, then a and c are not connected. So we may flip the ac-chain of a to free up the color c.



Configurations

Configurations are planar graphs.

Definition

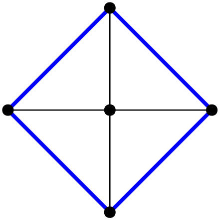
A configuration $\mathcal{C}$ is *contained* in a connected planar graph G if $G \setminus \mathcal{C}$ is connected.

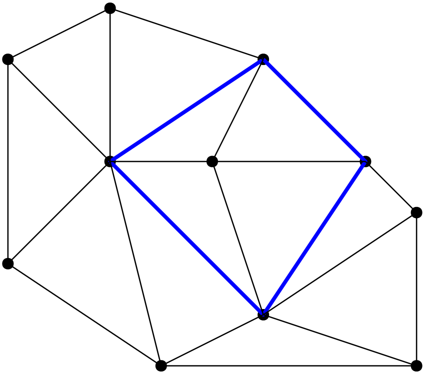
Definition

A configuration $\mathcal{C}$ is *reducible* in a graph G if its presence implies that the 4-coloring of G can be reduced to the 4-coloring of G' with less vertices.

For the five color theorem, vertices of $\deg(v) \leq 5$ are all reducible configurations. Therefore we can always break down a 5-coloring problem.

Configurations

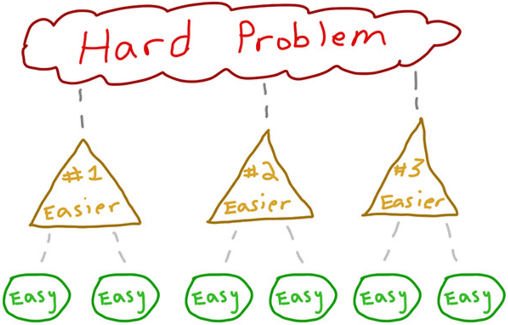




Fundament of the Four Color Theorem

Four Color Theorem

Every planar graph G strongly-contains a configuration $\mathcal{C}$ that is either k-reducible, D-reducible or C-reducible in G.



Rings

Rings play a key role in seperating a graph in two pieces with a common border.

Definition

A *ring* of n vertices R_n in a planar graph G is an induced cycle of G.

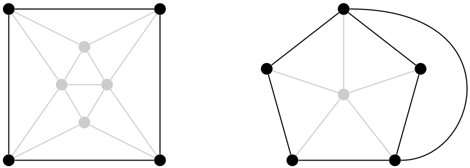


Figure: Valid ring (left). Invalid ring (right).

Ring colorings

Definition

The set of all 4-colorings of a ring R in a planar graph G is given by $\Phi(R \subset G)$ or $\Phi(G)$ if R is clear from the context.

Definition

The set of all ring colorings of R_n is given by $\Phi(n) = \Phi(R_n)$.

Rings are reducible

A ring on its own is a reducible configuration.

Theorem

The ring R_n with $n \geq 4$ is reducible in every planar graph G.

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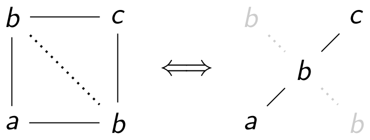
Proof.

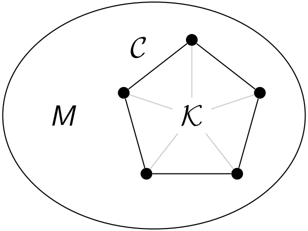
Because R_n is contained, the interior of R_n is empty. Contract two non-neighboring vertices v_1 and v_3 to a new vertex u. We obtain the smaller graph G'. To reverse a coloring of G', we give v_1 and v_3 the same color as u. We obtained a coloring for G. □



Ring configurations

Definition

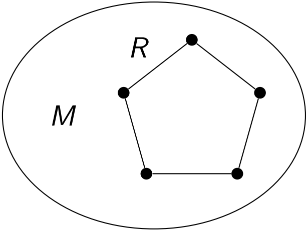
A planar graph $\mathcal{C} = R_n + \mathcal{K}$ consisting of a ring R_n and an interior $\mathcal{K}$ is called a *ring configuration* on R_n. $\mathcal{K}$ is called the *core* of $\mathcal{C}$.

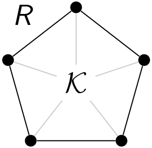


We have already shown reducibility if $\mathcal{K} = \varnothing$ (nothing)!

Finding common ring colorings

We split our graph into $M + R$ and $\mathcal{K} + R$. Now we try to find a common coloring on the ring R.





Such a coloring might not always be guaranteed.

Reducers

We can add extra vertices on the other side of the ring. This restricts the possible colorings we can encounter.

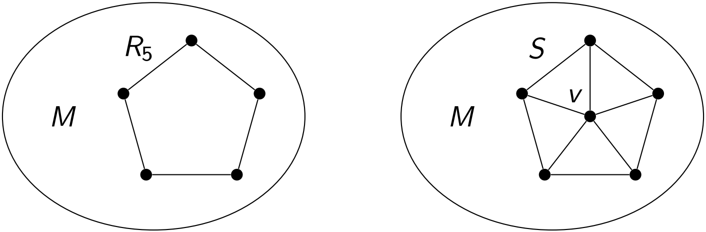


Figure: A graph that supports all ring colorings (left). A graph that only supports 3-colorings (right).

The ring + extra vertices and edges is a *reducer* graph.

Reducers

We restrict the size of the reducers such that $|S - R| \leq k$. By increasing k, we get more control but a larger graph to color.

Reducer Size

- If $k = 0$, then $S = R$ and we reduce to $M + R$ and $\mathcal{K} + R$. This is the smallest possible reduction and ideal situation.
- If $k = 1$, then we can set $S = R + v$. Our reduced graph is then one vertex larger.
- If $k \geq |\mathcal{C}|$, then there is no point in reducing, since we obtain the same graph $M + \mathcal{C}$.

k-Reducibility

Definition

A configuration $\mathcal{C} = \mathcal{K} + R$ is *k-reducible* if for all planar graphs $G = M + \mathcal{C}$ and some reducers S and S' on $\leq k < |\mathcal{C}|$ vertices, there exists a common ring coloring for $M + S$ and $\mathcal{K} + S'$.

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Definition

The ring R_n is *k-reducible* if all configurations $\mathcal{C}$ on R_n are at most k-reducible.

k-reducibility

Example

The rings R_2 and R_3 are 0-reducible. The only colorings are ab and abc. Therefore, there is always a common ring coloring for $M + R$ and $\mathcal{K} + R$.

We will be proving the 0-reducibility of R_4 and the 1-reducibility of R_5 after introducing Kempe-chains.

Kempe-chains

Definition

Let $G_{ab}(x)$ be the subgraph consisting of all the vertices colored ab in the coloring x of G. Then the *Kempe-chain* $\kappa_{ab}(v)$ or *ab-chain* of the vertex v is the component of $G_{ab}(x)$ that contains v.

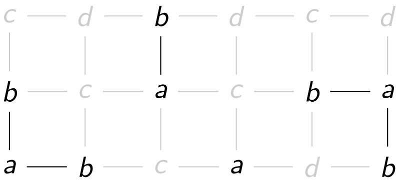


Figure: We write $v_1 \overset{ab}{\frown} v_2$ to indicate v_1 and v_2 lie on the same ab-chain.

Ring schemes

Definition

Given a coloring x of a planar graph G and the colors on its ring $x(R)$. The *scheme* on R of x consists of $x(R)$ with knowledge whether $u \in \kappa_{ab}(v)$ for two ring vertices $u, v \in R$ and colors ab.

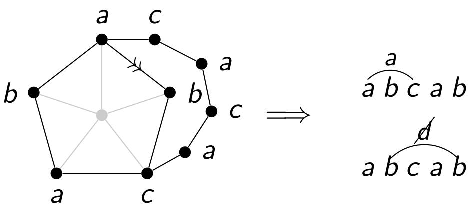


Figure: The edge with $\gg$ indicates order of vertices in the coloring.

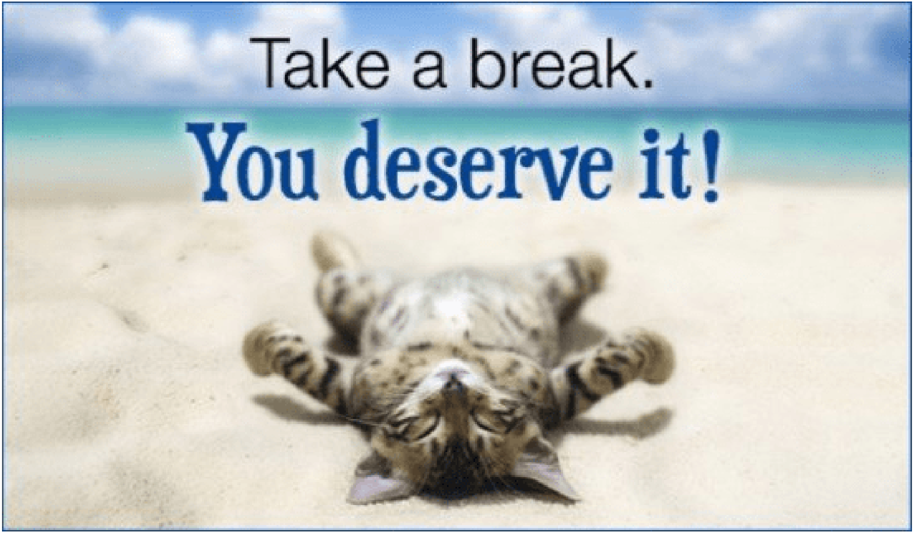
Schemes imply new colorings

Definition

Given two schemes x and y. We say that x implies y if $x = y$ or y can be obtained from x by flipping a Kempe-chain. Write $x \implies y$.

$$\overset{a}{\overparen{a\ b\ c}}\ a\ b \implies \overset{a}{\overparen{a\ b\ c}}\ a\ \mathbf{d}\,, \qquad a\ \overset{\cancel{d}}{\overparen{b\ c\ a\ b}} \implies a\ \overset{\cancel{d}}{\overparen{\mathbf{d}\ c\ a\ b}}\,.$$

This is the key argument we used in the five color theorem!



We will be back in 5 minutes.

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Theorem

The ring R_4 is 0-reducible

0-reducibility of ring R_4

Theorem

The ring R_4 is 0-reducible

Proof. We will show that there is a common ring coloring for any $M + R_4$ and $\mathcal{C}$. Let the ring colorings of the two graphs be given by

$$\mathrm{I} = \Phi(M + R_4) \quad \text{and} \quad \mathrm{II} = \Phi(\mathcal{C}). \tag{1}$$

0-reducibility of ring R_4

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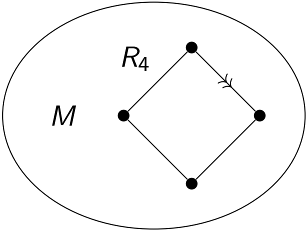
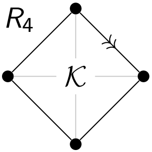
The situation is sketched below.





0-reducibility of ring R_4

Both parts have a plain ring R_4, which we have shown to be reducible. Therefore, we may contract any two opposing vertices.

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This gives us a guarantee on two colorings in both sets **I** and **II**.

$$\left\{\begin{matrix} abab \text{ or } abac, \\ baba \text{ or } baca \end{matrix}\right\} \subset I, II. \tag{2}$$

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$$\left\{\begin{matrix} abab & \text{or} & abac, \\ baba & \text{or} & baca \end{matrix}\right\} \subset I, II. \tag{2}$$

Therefore, we have 4 possible options for guaranteed colorings in I and II, 3 of which are unique.

$$1 = \{abab\}, \quad 2 = \left\{\begin{matrix} abab \\ baca \end{matrix}\right\}, \quad 3 = \left\{\begin{matrix} abac \\ baba \end{matrix}\right\}, \quad 4 = \left\{\begin{matrix} abac \\ baca \end{matrix}\right\}. \tag{3}$$

0-reducibility of ring R_4

$$① = \{abab\}, \quad ② = \begin{Bmatrix} abab \\ baca \end{Bmatrix}, \quad ③ = \begin{Bmatrix} abac \\ baba \end{Bmatrix}, \quad ④ = \begin{Bmatrix} abac \\ baca \end{Bmatrix}.$$

We consider all pairs that can occur for I and II. All but one pair already have a common ring coloring.

0-reducibility of ring R_4

$$① = \{abab\}, \quad ② = \begin{Bmatrix} abab \\ baca \end{Bmatrix}, \quad ③ = \begin{Bmatrix} abac \\ baba \end{Bmatrix}, \quad ④ = \begin{Bmatrix} abac \\ baca \end{Bmatrix}.$$

We consider all pairs that can occur for I and II. All but one pair already have a common ring coloring.

The only pair that does not directly have a common coloring is ① and ④.

0-reducibility of ring R_4

Let $\{abab\} \subset \mathrm{I}$ and $\{abac, baca\} \subset \mathrm{II}$ be guaranteed colorings.

0-reducibility of ring R_4

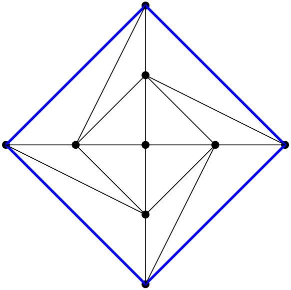
Let $\{abab\} \subset \mathrm{I}$ and $\{abac, baca\} \subset \mathrm{II}$ be guaranteed colorings.

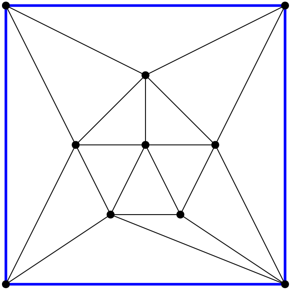
Suppose that in the coloring $\mathrm{I}(abab)$ we have $v_1 \overset{ad}{\frown} v_3$.

$$\begin{aligned} \mathrm{I}(abab) &= \overset{d}{\overbrace{a\ b\ a}}\ b \implies \mathrm{II}(abac) \\ \mathrm{I}(abab) &= \overset{\cancel{d}}{\overbrace{a\ b\ a}}\ b \implies \mathrm{I}(abcb) = \mathrm{II}(baca). \end{aligned} \tag{4}$$

Therefore, the guaranteed colorings always lead to a common coloring $\square$.

Examples of reducible configurations on R_4





Because R_4 is 0-reducible, the interior of **any** configuration on R_4 can be removed.

1-reducibility of ring R_5

Theorem

The ring R_5 is 1-reducible

1-reducibility of ring R_5

Theorem

The ring R_5 is 1-reducible

Proof. We will show that there is a common ring coloring for any $M + S$ and $\mathcal{K} + S'$.

Let the ring colorings of the two graphs again be given by

$$\mathrm{I} = \Phi(M + S) \quad \text{and} \quad \mathrm{II} = \Phi(\mathcal{K} + S'). \tag{5}$$

We can choose to reduce with any S and S' on ≤ 1 extra vertices. Each choice gives us guaranteed colorings for I and II.

1-reducibility of ring R_5

First we examine the guaranteed colorings without using a reducer.

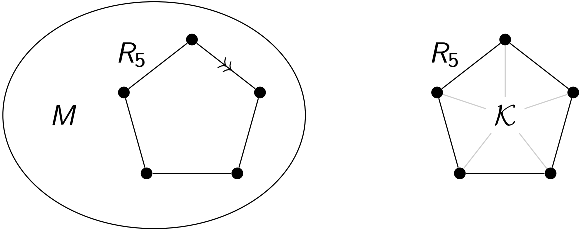


Figure: The reductions $M + R_5$ and $\mathcal{K} + R_5$.

These plain rings R_5 in each graph may be further reduced by contracting two opposing vertices.

1-reducibility of ring R_5

We can contract two opposing vertices of R_5 in 5 different ways. Each choice guarantees a coloring where two vertices are colored the same.

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$$\Phi^\star = \{a{*}{*}a{*}, \quad {*}a{*}{*}a, \quad a{*}a{*}{*}, \quad {*}a{*}a{*}, \quad {*}{*}a{*}a{*}\}. \tag{6}$$

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The $*$-colors are still unknown, but we are guaranteed that the colorings are possible, therefore $\Phi^\star \subset \mathrm{I}, \mathrm{II}$.

1-reducibility of ring R_5

Next, we examine the guaranteed colorings with a reducer that has 1 extra vertex.

1-reducibility of ring R_5

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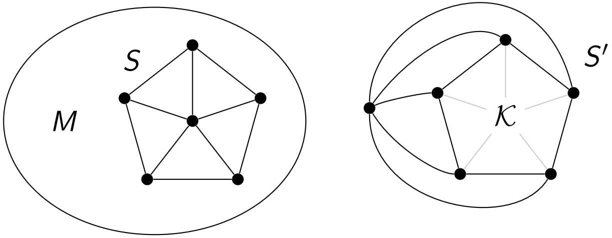

Figure: The reductions $M + S$ and $\mathcal{K} + S'$.

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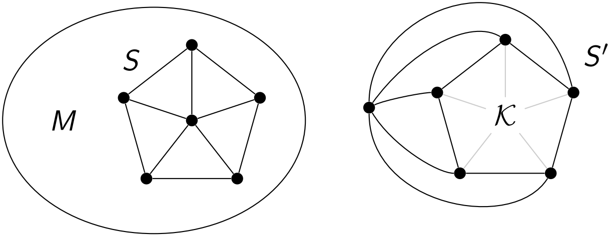

Figure: The reductions $M + S$ and $\mathcal{K} + S'$.

These reducers guarantee **one** 3-coloring of the ring.

1-reducibility of ring R_5

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$$\Phi^S = \{\underline{c}abab \textbf{ or } a\underline{c}bab \textbf{ or } ab\underline{c}ab \textbf{ or } aba\underline{c}b \textbf{ or } abab\underline{c}\}. \tag{7}$$

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These reducers guarantee **one** 3-coloring of the ring R_5.

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In total, we are guaranteed of two sets of colorings for I and II.

$$\Phi^\star, \Phi^S \;\subset \mathrm{I}, \mathrm{II}.$$

Next, we show that these guaranteed colorings are sufficient to find a common coloring in I and II.

1-reducibility of ring R_5

The 3-colorings are important because of two key properties that they have.

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Definition

The uniquely-colored vertex of a 3-coloring of R_5 is called the *marked vertex*, indicated by an underline such as in $\underline{c}abab$.

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Definition

The uniquely-colored vertex of a 3-coloring of R_5 is called the *marked vertex*, indicated by an underline such as in $\underline{c}abab$.

Definition

Two 3-colorings of R_5 are called *adjacent* if they have adjacent marked vertices, such as in $\underline{c}abab$ and $a\underline{c}bab$.

1-reducibility of ring R_5

Both sets I and II are guaranteed to have one 3-coloring from Φ^S. There are three cases that can occur.

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1. I and II have an adjacent coloring ($\underline{c}abab$ and $a\underline{c}bab$).

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1. I and II have an adjacent coloring ($\underline{c}abab$ and $a\underline{c}bab$).
2. I and II have a non-adjacent coloring ($\underline{c}abab$ and $ab\underline{c}ab$).

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Both sets I and II are guaranteed to have one 3-coloring from Φ^S. There are three cases that can occur.

1. I and II have an adjacent coloring ($\underline{c}abab$ and $a\underline{c}bab$).
2. I and II have a non-adjacent coloring ($\underline{c}abab$ and $ab\underline{c}ab$).
3. I and II have a coloring with the same marked vertex ($\underline{c}abab$ and $\underline{d}cbcb$). These are already equal, so we are done.

1-reducibility of ring R_5

1. I and II have an adjacent coloring ($\underline{c}abab$ and $a\underline{c}bab$).
2. I and II have a non-adjacent coloring ($\underline{c}abab$ and $ab\underline{c}ab$).

We have two lemmas to deal with these cases. Together they guarantee a common coloring in I and II.

$$\begin{array}{llll} ② \implies ① & \text{or} & \text{common coloring} & \text{(Lemma 1)} \\ \quad\quad\quad \downarrow & & & \\ \quad\quad\quad ① \implies & & \text{common coloring} & \text{(Lemma 2)} \end{array}$$

1-reducibility of ring R_5

Lemma 1

If I and II have a non-adjacent coloring, then they either have an adjacent coloring or a common coloring.

The second case results in a coloring adjacent to I($\underline{c}abab$) as desired.

1-reducibility of ring R_5

Lemma 1

If I and II have a non-adjacent coloring, then they either have an adjacent coloring or a common coloring.

Proof. Assume we have two non-adjacent colorings I($\underline{c}abab$) and II($ab\underline{c}ab$). Suppose that $v_3 \overset{bc}{\frown} v_5$ in II($ab\underline{c}ab$). This leads to

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1-reducibility of ring R_5

Lemma 1

If I and II have a non-adjacent coloring, then they either have an adjacent coloring or a common coloring.

Proof. Assume we have two non-adjacent colorings $\mathrm{I}(\underline{c}abab)$ and $\mathrm{II}(ab\underline{c}ab)$. Suppose that $v_3 \overset{bc}{\frown} v_5$ in $\mathrm{II}(ab\underline{c}ab)$. This leads to

$$
\begin{aligned}
\mathrm{II}(ab\underline{c}ab) &= a\ b\ \overset{b}{\widehat{c\ a\ b}} \implies \mathrm{II}(abcdb), \\
\mathrm{II}(ab\underline{c}ab) &= a\ b\ \overset{\cancel{b}}{\widehat{c\ a\ b}} \implies \mathrm{II}(a\underline{c}bab).
\end{aligned}
\tag{8}
$$

The second case results in a coloring adjacent to $\mathrm{I}(\underline{c}abab)$ as desired.

1-reducibility of ring R_5

The first case results in $\mathrm{II}(abcdb)$.

1-reducibility of ring R_5

The first case results in II($abcdb$). Consider the coloring I($*b**b$).

1-reducibility of ring R_5

The first case results in $\mathrm{II}(abcdb)$. Consider the coloring $\mathrm{I}(*b**b)$.

The two adjacent $*$-colors must be different from each other and b, therefore we may assume that we have $\mathrm{I}(*bcdb)$. The last $*$-color reveals 3 possibilities.

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The two adjacent $*$-colors must be different from each other and b, therefore we may assume that we have $\mathrm{I}(*bcdb)$. The last $*$-color reveals 3 possibilities.

$$\begin{array}{lcl} \mathrm{I}(abcdb) & = & \mathrm{II}(abcdb) \text{ from case 1,} \\ \mathrm{I}(cbc\underline{d}b) & \text{adjacent to} & \mathrm{II}(ab\underline{c}ab), \\ \mathrm{I}(db\underline{c}db) & = & \mathrm{II}(ab\underline{c}ab). \end{array} \tag{9}$$

1-reducibility of ring R_5

The first case results in $\mathrm{II}(abcdb)$. Consider the coloring $\mathrm{I}(*b{*}{*}b)$.

The two adjacent $*$-colors must be different from each other and b, therefore we may assume that we have $\mathrm{I}(*bcdb)$. The last $*$-color reveals 3 possibilities.

$$\begin{array}{lcl} \mathrm{I}(abcdb) & = & \mathrm{II}(abcdb) \text{ from case 1,} \\ \mathrm{I}(cbc\underline{d}b) & \text{adjacent to} & \mathrm{II}(ab\underline{c}ab), \\ \mathrm{I}(db\underline{c}db) & = & \mathrm{II}(ab\underline{c}ab). \end{array} \tag{9}$$

Therefore we obtain either a common coloring or an adjacent coloring $\square$.

1-reducibility of ring R_5

Lemma 2

If I and II have an adjacent coloring, then they have a common coloring.

1-reducibility of ring R_5

Lemma 2

If I and II have an adjacent coloring, then they have a common coloring.

Proof. Assume we have two adjacent colorings I($\underline{c}abab$) and II($a\underline{c}bab$). Suppose that $v_3 \overset{bd}{\frown} v_5$ in II($a\underline{c}bab$). This leads to

1-reducibility of ring R_5

Lemma 2

If I and II have an adjacent coloring, then they have a common coloring.

Proof. Assume we have two adjacent colorings $\mathrm{I}(\underline{c}abab)$ and $\mathrm{II}(a\underline{c}bab)$. Suppose that $v_3 \overset{bd}{\frown} v_5$ in $\mathrm{II}(a\underline{c}bab)$. This leads to

$$\begin{aligned}
\mathrm{II}(a\underline{c}bab) &= \; a\; c\; \overset{d}{\overparen{b\; a\; b}} \implies \mathrm{I}(\underline{c}abab). \\
\mathrm{II}(a\underline{c}bab) &= \; a\; c\; \overset{\cancel{d}}{\overparen{b\; a\; b}} \implies \mathrm{II}(acdab).
\end{aligned} \tag{10}$$

The first case leads to a common coloring as desired

1-reducibility of ring R_5

The second case results in II($acdab$). Consider the coloring I($a{*}{*}a{*}$).

1-reducibility of ring R_5

The second case results in II($acdab$). Consider the coloring I($a{*}{*}a{*}$).

We may again assume to have I($acda*$). Then we have 3 remaining possibilities for the $*$-color.

1-reducibility of ring R_5

The second case results in $\mathrm{II}(acdab)$. Consider the coloring $\mathrm{I}(a{*}{*}a{*})$.

We may again assume to have $\mathrm{I}(acda{*})$. Then we have 3 remaining possibilities for the $*$-color.

$$\begin{array}{lcll} \mathrm{I}(acdab) & = & & \mathrm{II}(acdab) \text{ from case 2}, \\ \mathrm{I}(ac\underline{d}ac) & = & \text{shifted } +2 & \mathrm{I}(\underline{c}abab), \\ \mathrm{I}(a\underline{c}dad) & = & & \mathrm{II}(a\underline{c}bab). \end{array}$$

Only the second case does not lead to a common coloring.

1-reducibility of ring R_5

We can repeat the same argument to continuously shift the marked vertex $+2$ to the right for **I** and **II**. This results in a pattern.

1-reducibility of ring R_5

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1-reducibility of ring R_5

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	v_1	v_2	v_3	v_4	v_5	v_1
1	I	II				I
2		II	I			
3			I	II		
4				II	I	
5					I	II

1-reducibility of ring R_5

We can repeat the same argument to continuously shift the marked vertex $+2$ to the right for I and II. This results in a pattern.

	v_1	v_2	v_3	v_4	v_5	v_1
1	I	II				I
2		II	I			
3			I	II		
4				II	I	
5					I	II

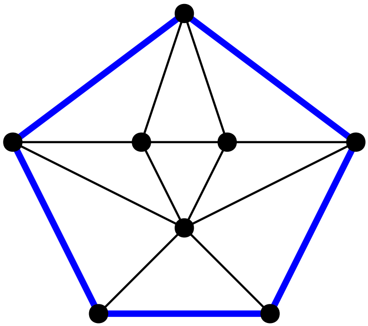
At iteration 5, we obtain that II and I have the same marked vertex v_1. Therefore, by repetition we finally obtain that

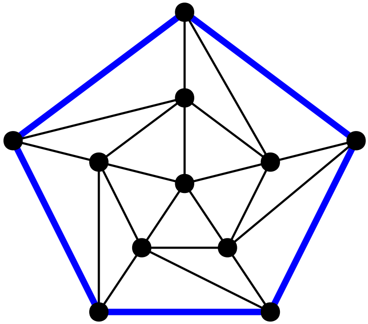
$$\mathrm{II}(a\underline{c}bab) \to \mathrm{II}(aba\underline{c}b) \to \mathrm{II}(\underline{c}abab) = \mathrm{I}(\underline{c}abab) \quad \square. \tag{11}$$

1-reducibility of ring R_5

Lemma 1 and Lemma 2 together guarantee a common ring coloring. This finishes the proof $\square$.

Examples of reducible configurations on R_5.





Example of non-reducible configurations on R_5.

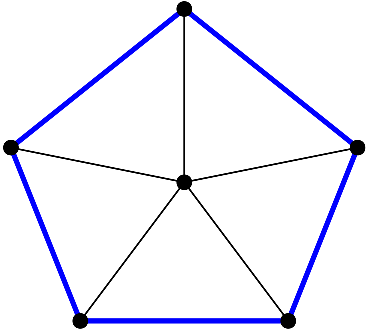


Figure: Replacing this configuration by our reducer yields the same graph. Therefore, this configuration does not fall under the 1-reducibility of R_5. If it were, then the four color theorem would proven.

We will be back in 5 minutes.

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The Birkhoff Diamond

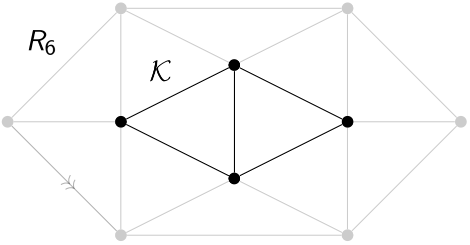

Figure: The Birkhoff Diamond (*bir*).

The degrees of vertices in the core $\mathcal{K}$ fully determines the ring around it.

The Birkhoff Diamond

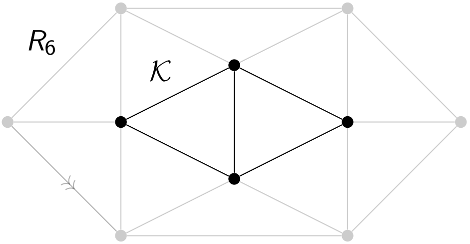

Figure: The Birkhoff Diamond (*bir*).

The degrees of vertices in the core $\mathcal{K}$ fully determines the ring around it.

$$\begin{array}{ccc} \bullet & \longrightarrow & \deg(v) = 5 \\ \cdot & \longrightarrow & \deg(v) = 6 \\ \circ & \longrightarrow & \deg(v) = 7 \\ \square & \longrightarrow & \deg(v) = 8 \end{array}$$

The Birkhoff Diamond

$\Phi(6)$			
ababab	abacbd	abcadc	abcdab
ababac	abacdb	abcbab	abcdac
ababcb	abacdc	abcbac	abcdad
ababcd	abcabc	abcbad	abcdbc
abacab	abcabd	abcbcb	abcdbd
abacac	abcacb	abcbcd	abcdcb
abacad	abcacd	abcbdb	abcdcd
abacbc	abcadb	abcbdc	
31			

Figure: All unique ring colorings of R_6. The colorings of $\Phi(\text{Bir}\Diamond)$ in green.

Implied colorings

Consider a chain $v_4 \overset{bd}{\frown} v_6$ in *ababab*.

$$
\begin{aligned}
a\ b\ a\ b\ \overset{d}{\frown}\ a\ b &\implies \boxed{\text{ababcb}} \\
a\ b\ a\ b\ \overset{\cancel{d}}{\frown}\ a\ b &\implies ababad = \boxed{\text{ababac}}.
\end{aligned} \tag{12}
$$

Implied colorings

Consider a chain $v_4 \overset{bd}{\frown} v_6$ in $ababab$.

$$\begin{aligned} a\ b\ a\ \overset{d}{\widehat{b\ a}}\ b \quad &\Longrightarrow \quad \boxed{\mathrm{ababcb}} \\ a\ b\ a\ \overset{\cancel{d}}{\widehat{b\ a}}\ b \quad &\Longrightarrow \quad ababad = \boxed{\mathrm{ababac}}. \end{aligned} \tag{12}$$

Therefore, the coloring $ababab$ implies the set

$$ababab \Longrightarrow \{ababcb,\ ababac\} \subset \Phi(\mathrm{Bir}\Diamond). \tag{13}$$

We say that $ababab$ is a *fixable* ring coloring of Bir$\Diamond$.

Implied colorings

Definition

A coloring x implies a set of colorings II if every scheme $x^\star$ of x implies a coloring $y \in \mathrm{II}$. Write $x \implies \mathrm{II}$.

Implied colorings

Definition

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Definition

A set of colorings I implies II if every $x \in \mathrm{I}$ implies II. Write $\mathrm{I} \implies \mathrm{II}$.

Implied colorings

$$\Phi_5(\mathrm{Bir}\Diamond) \implies \Phi_4(\mathrm{Bir}\Diamond) \implies \Phi_3(\mathrm{Bir}\Diamond) \implies \ldots \implies \Phi_0(\mathrm{Bir}\Diamond). \quad (14)$$

Implied colorings

$$\Phi_5(\mathrm{Bir}\Diamond) \implies \Phi_4(\mathrm{Bir}\Diamond) \implies \Phi_3(\mathrm{Bir}\Diamond) \implies \dots \implies \Phi_0(\mathrm{Bir}\Diamond). \quad (14)$$

Definition

A set of colorings I n-implies a set II if there exist sets B_i for $0 < i < n$ such that $I \implies B_{n-1}$, $B_i \implies B_{i-1}$ and $B_1 \implies$ II. We write I $\stackrel{n}{\implies}$ II.

Implied colorings

$$\Phi_5(\mathrm{Bir}\lozenge) \implies \Phi_4(\mathrm{Bir}\lozenge) \implies \Phi_3(\mathrm{Bir}\lozenge) \implies \ldots \implies \Phi_0(\mathrm{Bir}\lozenge). \quad (14)$$

Definition

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Example

$$\Phi_5(\mathrm{Bir}\lozenge) \overset{5}{\implies} \phi_0(\mathrm{Bir}\lozenge) \quad (15)$$

Implied colorings

$\Phi_0(\mathrm{Bir}\Diamond)$		Φ_1	Φ_2	Φ_3	Φ_4	Φ_5
ababac	abcadb	ababab	abacad	abacbd	abcabd	abcabc
ababcb	abcbab	ababcd	abcbdb	abcbdc	abcadc	
abacac	abcbac	abacab		abcdac	abcdbc	
abacbc	abcbad	abcbcb		abcdbd		
abacdb	abcbcd	abcdad				
abacdc	abcdab					
abcacb	abcdcb					
abcacd	abcdcd					
16		5	2	4	3	1

Figure: All *n*-implying sets of Bir$\Diamond$. Together a total of 31 colorings. Only differences are shown.

D-reducibility

Definition

The *max-implying* set $\overline{\Phi}(\mathcal{C})$ of a configuration $\mathcal{C}$ is the largest n-implying set $\Phi_n(\mathcal{C})$.

D-reducibility

Definition

The *max-implying* set $\overline{\Phi}(\mathcal{C})$ of a configuration $\mathcal{C}$ is the largest n-implying set $\Phi_n(\mathcal{C})$.

Definition

A configuration $\mathcal{C}$ on R_n is D-reducible if $\overline{\Phi}(\mathcal{C}) = \Phi(n)$.

D-reducibility

Definition

The *max-implying* set $\overline{\Phi}(\mathcal{C})$ of a configuration $\mathcal{C}$ is the largest n-implying set $\Phi_n(\mathcal{C})$.

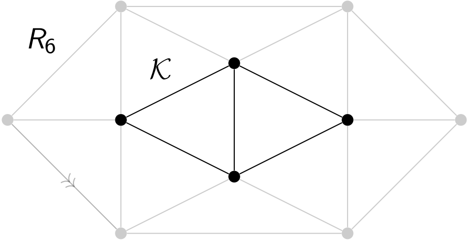
Definition

A configuration $\mathcal{C}$ on R_n is D-reducible if $\overline{\Phi}(\mathcal{C}) = \Phi(n)$.

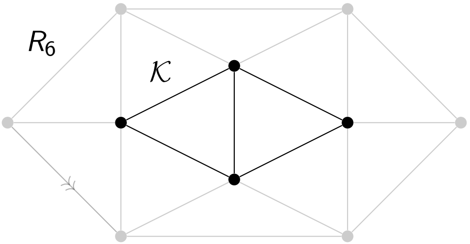
Example

Because $\Phi_5(\mathrm{Bir}\Diamond) = \Phi(6)$, Bir$\Diamond$ is D-reducible.

D-reducibility



D-reducibility



What can we do if a ring coloring **can not** be fixed?

The Bernhart Diamond

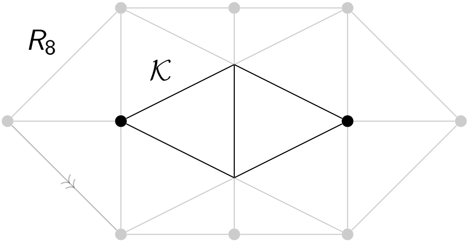


Figure: The Bernhart Diamond (Ber$\Diamond$). An example of a non D-reducible configuration. Why?

C-reducibility

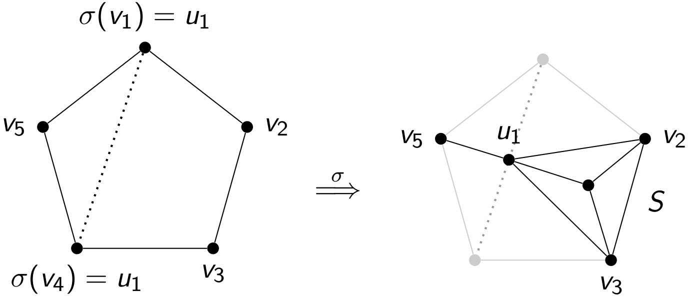
- valid
- fixable
- reducer fixable
- reducer unfixable
- symmetry fault
- unfixable

abababab	abacadac	abcababc	abcadbab	abcbcbdb	abcdadbc
abababac	abacadad	abcababd	abcadbac	abcbcbdc	abcdadbd
abababcb	abacadbc	abcabacb	abcadbad	abcbcdab	abcdadcb
abababcd	abacadbd	abcabacd	abcadbcb	abcbcdac	abcdadcd
ababacab	abacadcb	abcabadb	abcadbcd	abcbcdad	abcdbabc
ababacac	abacadcd	abcabadc	abcadbdb	abcbcdbc	abcdbabd
ababacad	abacbabc	abcabcab	abcadbdc	abcbcdbd	abcdbacb
ababacbc	abacbabd	abcabcac	abcadcab	abcbcdcb	abcdbacd
ababacbd	abacbacb	abcabcad	abcadcac	abcbcdcd	abcdbadb
ababacdb	abacbacd	abcabcbc	abcadcad	abcbdabc	abcdbadc
ababacdc	abacbadb	abcabcbd	abcadcbc	abcbdabd	abcdbcab
ababcabc	abacbadc	abcabcdb	abcadcbd	abcbdacb	abcdbcac
ababcabd	abacbcab	abcabcdc	abcadcdb	abcbdacd	abcdbcad
ababcacb	abacbcac	abcabdab	abcadcdc	abcbdadb	abcdbcbc
ababcacd	abacbcad	abcabdac	abcbabab	abcbdadc	abcdbcbd
ababcadb	abacbcbc	abcabdad	abcbabac	abcbdbab	abcdbcdb
ababcadc	abacbcbd	abcabdbc	abcbabad	abcbdbac	abcdbcdc
ababcbab	abacbcdb	abcabdbd	abcbabcb	abcbdbad	abcdbdab
ababcbac	abacbcdc	abcabdcb	abcbabcd	abcbdbcb	abcdbdac
ababcbad	abacbdab	abcabdcd	abcbabdb	abcbdbcd	abcdbdad
ababcbcb	abacbdac	abcacabc	abcbabdc	abcbdbdb	abcdbdbc
ababcbcd	abacbdad	abcacabd	abcbacab	abcbdbdc	abcdbdbd
ababcbdb	abacbdbc	abcacacb	abcbacac	abcbdcab	abcdbdcb
ababcbdc	abacbdbd	abcacacd	abcbacad	abcbdcac	abcdbdcd
ababcdab	abacbdcb	abcacadb	abcbacbc	abcbdcad	abcdcabc
ababcdac	abacbdcd	abcacadc	abcbacbd	abcbdcbc	abcdcabd
ababcdad	abacdabc	abcacbab	abcbacdb	abcbdcbd	abcdcacb
ababcdbc	abacdabd	abcacbac	abcbacdc	abcbdcdb	abcdcacd
ababcdbd	abacdacb	abcacbad	abcbadab	abcbdcdc	abcdcadb
ababcdcb	abacdacd	abcacbcb	abcbadac	abcdabab	abcdcadc
ababcdcd	abacdadb	abcacbcd	abcbadad	abcdabac	abcdcbab
abacabab	abacdadc	abcacbdb	abcbadbc	abcdabad	abcdcbac
abacabac	abacdbab	abcacbdc	abcbadbd	abcdabcb	abcdcbad
abacabad	abacdbac	abcacdab	abcbadcb	abcdabcd	abcdcbcb
abacabcb	abacdbad	abcacdac	abcbadcd	abcdabdb	abcdcbcd
abacabcd	abacdbcb	abcacdad	abcbcabc	abcdabdc	abcdcbdb
abacabdb	abacdbcd	abcacdbc	abcbcabd	abcdacab	abcdcbdc
abacabdc	abacdbdb	abcacdbd	abcbcacb	abcdacac	abcdcdab
abacacab	abacdbdc	abcacdcb	abcbcacd	abcdacad	abcdcdac
abacacac	abacdcab	abcacdcd	abcbcadb	abcdacbc	abcdcdad
abacacad	abacdcac	abcadabc	abcbcadc	abcdacbd	abcdcdbc
abacacbc	abacdcad	abcadabd	abcbcbab	abcdacdb	abcdcdbd
abacacbd	abacdcbc	abcadacb	abcbcbac	abcdacdc	abcdcdcb
abacacdb	abacdcbd	abcadacd	abcbcbad	abcdadab	abcdcdcd
abacacdc	abacdcdb	abcadadb	abcbcbcb	abcdadac	
abacadab	abacdcdc	abcadadc	abcbcbcd	abcdadad	

Reducers again

Reducers can be used to restrict the possible ring colorings. We need a reducer whose generated colorings are all fixable. We have already seen reducers when proving the 1-reducibility of R_5!

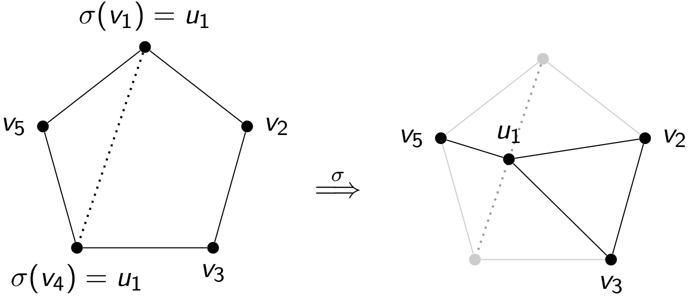
A reducer consists of a contraction and extra interior edges and vertices.



Reducers again

Definition

A ring contraction $\sigma(v)$ on R is a map from R to the contracted ring $\sigma(R)$. Neighboring vertices of R may not be mapped to the same vertex by σ.

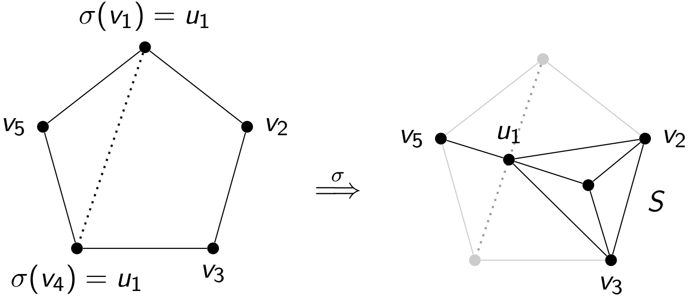


Reducers again

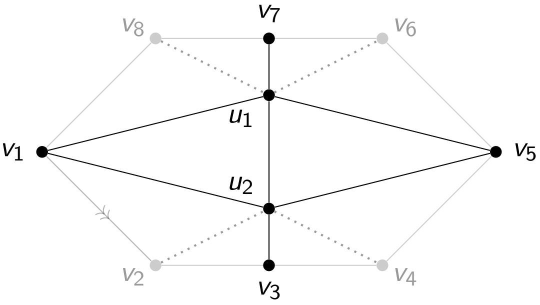
Definition

A reducer (S, σ) of a configuration $\mathcal{C}$ consists of

- A contraction $\sigma(v)$ on R.
- A graph $|S| < |\mathcal{C}|$ whose boundary is the contracted ring $\sigma(R)$.



Reducer for the Bernhart Diamond



The reducer generates the following type of colorings.

$$v_1 \; u_2 \; v_3 \; v_5 \; u_1 \; v_7 \quad \mapsto \quad v_1 \; u_2 \; v_3 \; u_2 \; v_5 \; u_1 \; v_7 \; u_1. \tag{16}$$

C-reducibility

Definition

A configuration $\mathcal{C}$ is C-reducible if $\Phi(S, \sigma) \subset \overline{\Phi}(\mathcal{C})$ for some reducer (S, σ).

Example

The Bernhart Diamond (Ber$\Diamond$) is C-reducible. The red colorings are *symmetry faults* that can be shown to be fixable

Symmetry Faults

The two red colorings generated by the reducer are not fixable. Does this mean that Ber$\Diamond$ is not C-reducible?

$$\begin{array}{ll} \text{abcbacbc} & (R1) \\ \text{abcbdcbc} & (R2) \end{array} \tag{17}$$

Symmetry Faults

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$$\begin{array}{ll} \text{abcbacbc} & (R1) \\ \text{abcbdcbc} & (R2) \end{array} \tag{17}$$

Bernhart has shown in 1947 that these colorings *can* in fact be fixed. So there must be an error in $\overline{\Phi}(\text{Ber}\Diamond)$.

Symmetry Faults

Consider the following colorings.

$$\begin{array}{llllll} \text{abcbacbc} & (R1) & \text{abcbadbc} & (F1) & \underline{\text{abcbacac}} & (G1) \\ \text{abcbdcbc} & (R2) & \text{abcbacbd} & (F1\star) & \underline{\text{abcbdbcb}} & (G2) \end{array} \tag{18}$$

Symmetry Faults

Consider the following colorings.

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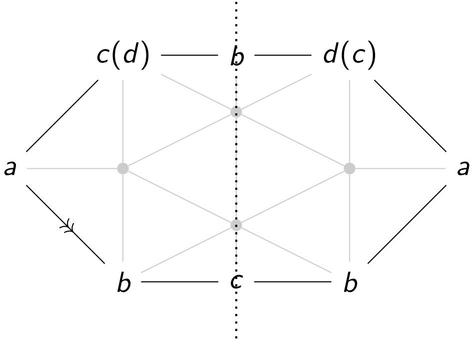
Then we have the implications

$$\begin{array}{lll} R2 \implies \{G2, & R1\} & \\ & \downarrow & \\ & R1 \implies \{G1, F1\} & \end{array} \tag{19}$$

Therefore, fixability of $R1$ and $R2$ depends fully on $F1$.

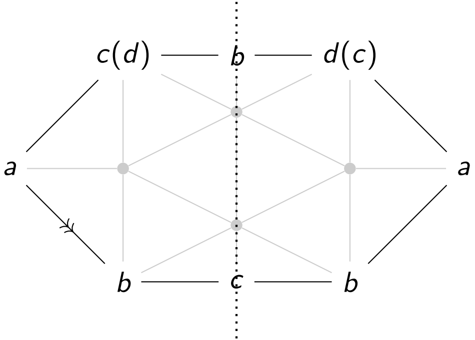
Symmetry Faults

abcbacbc	$(R1)$	abcbadbc	$(F1)$	<u>abcbacac</u>	$(G1)$	(20)
abcbdcbc	$(R2)$	abcbacbd	$(F1\star)$	<u>abcbdbcb</u>	$(G2)$	



Symmetry Faults

abcbacbc	$(R1)$	abcbadbc	$(F1)$	<u>abcbacac</u>	$(G1)$	(20)
abcbdcbc	$(R2)$	abcbacbd	$(F1\star)$	<u>abcbdbcb</u>	$(G2)$	



The colorings $F1$ and $F1\star$ are symmetric. Therefore, $F1$ must be fixable!

Definition

A graph symmetry of G is a bijection $f(v)$ on the vertices of G that preserves the neighbors of every vertex.

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A *symmetry fault* of a configuration $\mathcal{C}$ is a coloring x that is not in $\overline{\Phi}(\mathcal{C})$, but whose symmetry $x\star = f(x)$ is.

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Definition

A *symmetry fault* of a configuration $\mathcal{C}$ is a coloring x that is not in $\overline{\Phi}(\mathcal{C})$, but whose symmetry $x\star = f(x)$ is.

Bernhart mentioned the same problematic colorings in his proof. Is it a coincidence? We have not delved deeper into this problem.

Table of Contents

Conclusion

- k-reducibility extends upon ideas from the Five Color Theorem.
- D-reducibility extends upon k-reducibility for rings 6 and above.
- C-reducibility extends upon D-reducibility by avoiding bad ring colorings.
- Unavoidability guarantees that every planar graph has a reducible configuration.
- The Birkhoff Diamond and Bernhart Diamond illustrate D and C-reducibility.
- Symmetry faults in the Bernhart Diamond require more attention.

Conclusion

The more advanced the concept of reducibility, the less reducible configurations are needed (A/B-reducibility is a case of C reducibility).

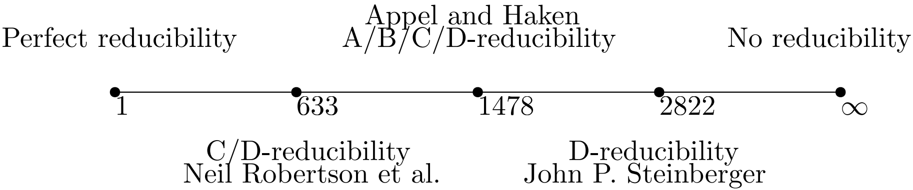

Figure: Number of reducible configurations on rings R_6 and higher in proofs of the four color theorem

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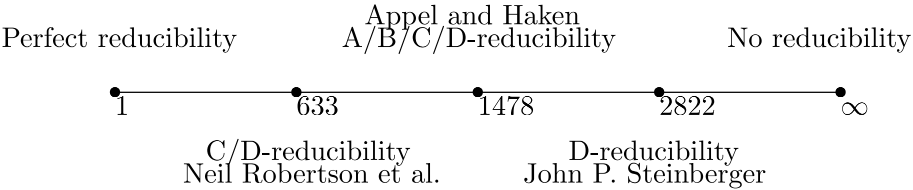

Figure: Number of reducible configurations on rings R_6 and higher in proofs of the four color theorem

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Does "Perfect reducibility" exist? If so, it will be the heart of the four color theorem.

The End

Thank you for your attention. Questions?